Modeling heterogeneous risk-taking behavior in route choice: A stochastic dominance approach

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A R T I C L E   I N F O

Keywords:
Route choice
Stochastic dominance
General dynamic programming
Risk aversion
Ruin aversion

A B S T R A C T

This paper proposes a unified approach to modeling heterogeneous risk-taking behavior in route choice based on the theory of stochastic dominance (SD). Specifically, the first-, second-, and third-order stochastic dominance (FSD, SSD, TSD) are respectively linked to insatiability, risk-aversion and ruin-aversion within the framework of utility maximization. The paths that may be selected by travelers of different risk-taking preferences can be obtained from the corresponding SD-admissible paths, which can be generated using general dynamic programming. This paper also analyzes the relationship between the SD-based approach and other route choice models that consider risk-taking behavior. These route choice models employ a variety of reliability indexes, which often make the problem of finding optimal paths intractable. We show that the optimal paths with respect to these reliability indexes often belong to one of the three SD-admissible path sets. This finding offers not only an interpretation of risk-taking behavior consistent with the SD theory for these route choice models, but also a unified and computationally viable solution approach through SD-admissible path sets, which are usually small and can be generated without having to enumerate all paths. A generic label-correcting algorithm is proposed to generate FSD-, SSD-, and TSD-admissible paths, and numerical experiments are conducted to test the algorithm and to verify the analytical results.

1. Introduction

Existing behavioral studies have revealed the significant influence of travel time reliability on travelers’ route choice decisions (e.g., Abdel-Aty et al., 1995; Lam and Small, 2001; Liu et al., 2004). Two dimensions of travel reliability are concerned in this paper. The first dimension is the probability of completing a trip within a given time budget, the so-called on-time arrival probability. This measure is related to how the decision maker determines the importance of a trip. For instance, a desperate job hunter may wish to arrive on time with 99% probability for a job interview whereas the same person may not care as much about reliability on a casual trip to a coffee shop. The second dimension has to do with the fact that for the trip deemed as equally important (i.e. the same on-time arrival probability is required), two individuals may choose different routes and reserve different amounts of time for travel depending on their risk-taking preference. It is worth noting that, while determining the importance of a trip depends on many psychological and behavioral factors, this process does not involve taking risks in its own right. In other words, risk-taking occurs only when decisions (e.g., route, departure time) are made to accomplish the desired goal, i.e., arriving on-time with a given probability, not when the goal is set. To some extent the above dichotomy is related to the reliability/unreliability aspects discussed in Chen and Zhou (2009), although the interpretation differs here.

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1.1. Literature review

This paper considers the circumstance where travelers have to choose the best from a set of routes with random travel times. The distribution of these random travel times are given and subject to no perception or measurement errors. In this context, the simplest behavioral assumption for route choice is that travelers would always minimize the expected travel time. This assumption leads to numerous variants of optimal path problems (see e.g., Hall, 1986; Fu and Riley, 1998; Fu, 2001; Miller-Hooks and Mahmassani, 2000; Miller-Hooks, 2001; Waller and Ziliakopoulos, 2002; Fan et al., 2005; Gao and Chabini, 2006). However, it is easy to see that minimizing the expected travel time does not necessarily account for travel reliability. Route choice models that attempt to incorporate travel reliability can be grouped into the following four classes according to how the “reliability index” is defined.

1. On-time arrival probability/percentile travel time. In Frank (1969), travelers are assumed to maximize the probability of completing a trip within a given time, or equivalently, minimizing the percentile travel time (PTT) for a desired on-time arrival probability. Fan et al. (2005) applied the same definition to an adaptive routing problem. Nie and Wu (2009a) showed that the minimum PTT paths can be solved by finding all non-dominated paths under the first-order stochastic dominance (FSD) rule, and proposed an algorithm based on general dynamic programming. Nie and Wu (2009b) incorporated the correlations between travel times on adjacent links into the above problem. Using FSD to compare paths with random travel times was also explored in Miller-Hooks and Mahmassani (2003). Based on the route choice model proposed in Nie and Wu (2009a), Nie et al. (submitted for publication) conducted a case study of reliable route guidance using a Chicago regional transportation network.

2. Travel time budget/effective travel time. Hall (1983) noted that travelers tend to reserve a safety margin to hedge against variations of travel times. The sum of the mean travel time and this safety margin is called effective travel time (ETT) or travel time budget (TTB). The safety margin is the product of the standard deviation of travel time and a scalar called punctuality parameter. If travel times are normally distributed, the punctuality parameter has a one-to-one correspondence with the on-time arrival probability, in which case the effective travel time equals percentile travel time. However, this equivalence does not hold in general, as explained in Section 4. The concept of TTB has been integrated into shortest path problems (Sivakumar and Batta, 1994; Sen et al., 2001) and traffic assignment (Uchida and Iida, 1993; Lo et al., 2006; Shao et al., 2006). Chen and Zhou (2010) argued that TTB does not properly account for risk-taking behavior (or the unreliability aspect of travel time variability, as they put it). They postulated that travelers would reserve another “risk” margin called mean-excess travel time (METT) to avoid unacceptable disruptions to their schedule. METT is the conditional expectation of the travel time larger than TTB, which is also known as conditional value-at-risk (CVaR) or tail value-at-risk (Tail VaR) in finance and economics.

3. Expected utility theory. Expected utility theory postulates that an individual chooses an alternative to maximize the expected utility of returns. While various functional forms can be used to represent travelers’ utility in route choice, effective algorithms exist for the optimal path problem only for linear or exponential utility functions (Loui, 1983; Mirchandani and Soroush, 1987). To capture risk-averse behavior, Mirchandani and Soroush (1987) and Yin et al. (2004) assumed that travelers’ utility is a quadratic function of travel time, which however prevents one from using the standard shortest path algorithm to find the optimal path. Loui (1983) showed that the quadratic utility problem can be converted to a bi-criteria shortest path problem whose solution is a set of non-dominated paths obtained by simultaneously minimizing the mean and variance of path travel times. The mean–variance rule was first proposed by Markowitz (1952, 1970) for portfolio selection in finance, and has since been extensively used to compare random variables in other fields. Despite its popularity, the mean–variance rule is applicable only for the quadratic utility or in cases where the random variables follow normal distributions (Samuelson, 1970). For one thing, the quadratic utility implies increasing absolute risk aversion, which is inconsistent with the known economic phenomena (Arrow, 1971; Stiglitz, 1970). The assumption of normal distributions is often violated in real-world applications (Cootner, 1964; Li et al., 2006).

4. Robust optimization. In the framework of robust optimization, decision-makers are assumed to maximize return in the worst scenario. When this concept is applied in route choice, it implies finding the best route to minimize the worse-case travel time (Yu and Yang, 1998; Bell and Cassir, 2002). When correlations between travel time are considered, the robust optimal path problems are NP-hard even under restrictive assumptions (Yu and Yang, 1998). Bertsimas and Sim (2003) studied a robust optimization model for general network flow problems without considering correlations, and proposed a polynomial algorithm to find robust shortest paths. Ordonez and Stier-Moses (2010) applied Bertsimas’s robust shortest path algorithm to solve a robust traffic assignment problem.

1.2. Overview

This paper proposes to model risk-taking behavior in route choice using the theory of stochastic dominance (SD) (Hanoch and Levy, 1969; Hadar and Russell, 1971; Rothschild and Stiglitz, 1970; Whitmore, 1970), which has been extensively used in finance and economics to rank random variables when their distributions are known. The premise of the SD theory is to capture the common risk preferences of all individuals whose utility functions meet certain criteria. Specifically, the SD theory relates risk-taking behavior to the shape of individual’s utility function through utility maximization. For example, it can be shown that any risk-averse traveler has a decreasing and concave utility function.
(cf. Section 2.2.2), and that to maximize his/her utility, such a traveler would always prefer a random travel time $X$ to another random variable $Y$ if and only if $X$ dominates $Y$ by the second-order stochastic dominance. Similarly, insatiable and ruin aversion can be related to the first- and third-order stochastic dominance (cf. Sections 2.2.1 and 2.2.3). The paths that may be selected by travelers of different risk-taking preferences can be obtained by enumerating all non-dominated paths according to the corresponding SD rules, called SD-admissible paths hereafter. We shall show that general dynamic programming can be employed to generate such SD-admissible paths and subsequently can provide a generic label-correcting algorithm. While they are not unique, the admissible paths can be used as a basis for determining the optimal path once the utility function is given or other criteria are specified.

This paper also reveals the connection between the SD-based approach and the first three classes of reliability-based route choice models mentioned above. Note that these models define “reliability” using very different indexes. Importantly, utility function is given or other criteria are specified. Therefore, the SD-based approach provides a unified and computationally viable solution framework to reliability-based optimization. In most cases, path enumeration is considered unavoidable, even though it is computationally impractical except for very small problems. We shall show that the optimal solutions for these reliability-based route choice models often belong to one of the SD-admissible path sets. For one thing, this relationship links the preference for a reliability index to a risk-taking preference consistent with the interpretation by utility maximization. Perhaps more importantly, it suggests that these optimal paths can be found from an SD-admissible path set, which is relatively small and can be identified without having to enumerate all paths. The rest of this paper is organized as follows. Section 2 introduces the SD theory and shows how it characterizes risk-taking behavior in route choice. Section 3 introduces and characterizes the admissible path sets under various stochastic dominance rules. Section 4 analyzes the relationship between the SD-admissible path sets and the optimal paths obtained from five specific route choice models, which all belong to one of the first three classes of models reviewed in Section 1.1. Section 5 first establishes that the SD-admissible paths can be found based on general dynamic programming and then presents a generic label-correcting algorithm. Most implementation details of the algorithm, however, are omitted for brevity. Small and large numerical examples are provided in Section 6 to verify the analysis and to demonstrate the computational performance of the proposed algorithm. Section 7 concludes the paper.

2. Stochastic dominance (SD) theory

2.1. Preliminaries

The stochastic dominance (SD) theory is widely used to compare random variables according to their (known) distributions. In the conventional setting, the utility function is always assumed to be increasing; that is, decision makers always prefer more quantities of a random variable (e.g. the return of an investment). In the context of route choice where travel time is often a dominating decision variable, however, travelers’ utility typically decreases with travel time. Keeping this in mind and denoting $F_X$ as the cumulative distribution function (CDF) of random variable $X$, the first-, second-, and third-order SD are defined as follows.

**Definition 1.** (FSD $\succ_1$) A random variable $X$ dominates another random variable $Y$ in the first order, denoted as $X \succ_1 Y$, if $F_X(t) \geq F_Y(t), \forall t$, and $\exists$ at least an open interval $A \in [0, T]$ with nonzero Lebesgue measure such that $F_X(t) > F_Y(t), \forall t \in A$.

**Definition 2.** (SSD $\succ_2$) A random variable $X$ dominates another random variable $Y$ in the second order, denoted as $X \succ_2 Y$, if $\int_0^T F_X(w)dw \geq \int_0^T F_Y(w)dw, \forall t$, and $\exists$ at least an open interval $A \in [0, T]$ with nonzero Lebesgue measure such that $\int_0^T F_X(w)dw > \int_0^T F_Y(w)dw, \forall t \in A$.

**Definition 3.** (TSD $\succ_3$) A random variable $X$ dominates another random variable $Y$ in the third order, denoted as $X \succ_3 Y$, if $\int_0^T \int_0^T F_X(w)dw dt \geq \int_0^T \int_0^T F_Y(w)dw dt, \forall t \leq T$, and $\exists$ at least an open interval $A \in [0, T]$ with nonzero Lebesgue measure such that $\int_0^T \int_0^T F_X(w)dw dt > \int_0^T \int_0^T F_Y(w)dw dt, \forall t \in A$.

where $T$ is a finite upper bound of the support.

The following example demonstrates the concept of stochastic dominance.

**Example 1.** Consider four paths whose travel times are discrete random travel times. The cumulative probability functions (CDF) of these random travel times are plotted in Fig. 1. The details of these distributions (including probability mass functions, as well as the means, variances and skewness) are also reported in the figure. Table 1 shows the calculation needed to determine FSD, SSD and TSD for the four paths. For example, Path 4 is dominated by any other path in the first order
according to Definition 1 because for any \( t = 1, 2, \ldots, 9 \), the cumulative probability of Path 4 is always smaller than that of any other three paths. Also, Path 1 is dominated by Path 2 or Path 3 in the second order according to Definition 2 because for any \( t = 1, 2, 3 \), the area under the CDF curve of Path 3 is always smaller than that of Path 1. Finally, the reader can verify that Path 2 is dominated by Path 3 in the third order according to Definition 3.

2.2. SD and risk-taking behavior

A widely adopted behavioral assumption in economics and finance (e.g., Friedman and Savage, 1948) states that decision makers always choose the alternative that provides maximum expected utility. Specifically, if \( E[U(X)] > E[U(Y)] \), then \( X \) is preferred to \( Y \), where \( U \) is the utility function and \( E[\cdot] \) denotes the expectation operator. The SD relationship between two random variables can be interpreted within this framework, as shown below.

**Theorem 1.** A random variable \( X \) dominates another random variable \( Y \\
1. in the first order, i.e, \( X >_1 Y \), if and only if \( E[U(X)] > E[U(Y)] \) for any \( U \) such that \( U' < 0 \);
2. in the second order, i.e, \( X >_2 Y \), if and only if \( E[U(X)] > E[U(Y)] \) for any \( U \) such that \( U' < 0, U'' < 0; \) and
3. in the third order, i.e, \( X >_3 Y \), if and only if \( E[U(X)] > E[U(Y)] \) for any \( U \) such that \( U' < 0, U'' < 0, U''' < 0 \).

**Proof.** The results can be proven similarly as in Bawa (1975), but note that Bawa (1975) considers increasing utility functions. □

2.2.1. FSD and insatiability

A decision maker is insatiable if his/her utility is a strictly increasing or decreasing monotone function of the quantity of the random variable of interest. Whether the utility function is increasing or decreasing depends on whether the decision maker prefers more quantities of the random variable or not. In route choice, for example, higher travel time generally leads to lower utility. Thus, the utility function of an insatiable traveler is always decreasing. According to Theorem 1, FSD is the rule for all insatiable decision makers. That is, if \( X >_1 Y \), then any insatiable decision maker would prefer \( X \) to \( Y \). Note that FSD only establishes a partial order, and hence two paths are not always properly ranked under FSD. Suppose the decision maker’s choice set includes only Paths 1 and 2 in Example 1. It is clear that both paths are preferred by some insatiable travelers, but not by all of them, because the two paths do not dominate each other in the first order. According to Columns 2 and 3 in Table 1, if the traveler desires to arrive on-time with a probability of 0.5, Path 1 is a better choice because it gives a lower 50-percentile travel time (\( t = 5 \)). If the on-time arrival probability is 0.8, Path 2 becomes the winner of the two. It is worth emphasizing that the FSD rule does not properly incorporate risks, as explained below.

2.2.2. SSD and risk aversion

A decision maker is considered “risk-averse” in this paper if he/she always prefers the expectation of a random variable, i.e., \( E[X] \), to \( X \) itself (Friedman and Savage, 1948). Mathematically, this implies \( \delta_X \sim X \Rightarrow E[U(\delta_X)] > E[U(X)] \Rightarrow U'(X) > 0 \). According to Jensen’s inequality, the utility function \( U(\cdot) \) satisfies the above condition if and only if it is concave, i.e., \( U'' < 0 \). It follows from the second statement in Theorem 1 that \( X >_2 Y \) if and only if all risk-averse decision makers prefer \( X \) to \( Y \). Risk-averse decision makers are a sub-set of insatiable decision makers. We have shown in the above that an insatiable traveler’s choice depends on his/her desired on-time arrival probability, if Paths 1 and 2 in Example 1 form a complete choice set. However, recall that Path 2 dominates Path 1 in the second order according to Example 1. Therefore, any risk-averse traveler will always prefer Path 2 to Path 1 regardless of the desired on-time arrival probability. In this case, ranking the two paths by SSD not only embeds risk-averse behavior but also resolves the choice ambiguity.

2.2.3. TSD and ruin aversion

According to Heyer (2001) and Ullrich (2009), ruin-averse decision makers are “willing to accept a small, almost certain loss in exchange for the remote possibility of large returns”, and conversely, are “unwilling to accept a small, almost certain gain in exchange for the remote possibility of ruin”. When the utility function is decreasing, ruin aversion corresponds negative skewness in the probability density function (PDF) of a random variable. Fig. 2 plots the PDFs of random travel times of two paths, where PDF 1 is positively skewed and PDF 2 is negatively skewed. Let \( t_1 \) and \( t_4 \) be the mean and maximum realization of Path 1’s travel time, and let \( t_2 \) and \( t_3 \) be the mean and the maximum realization of Path 2’s travel time, respectively. A ruin-averse traveler prefers Path 2 to Path 1 even though Path 2 has a longer mean travel time (\( t_3 < t_2 \)), because \( t_3 < t_4 \). In other words, the traveler would like to accept a slightly longer average travel time in order to avoid encountering a very significantly delay.

The relationship between TSD and ruin aversion can be illustrated using the Taylor expansion of the expected utility (Heyer, 2001; Ullrich, 2009).
Recalling that the skewness is measured by $E[(X - E[X])^3]/r^3$, where $r$ is the standard deviation, $U'' < 0$ indicates that an expected-utility-maximization decision maker would always prefer negative skewness (or ruin aversion), everything else equal. According to Theorem 1, if $X$ dominates $Y$ in the third order, it implies that $X$ is preferred to $Y$ by all travelers whose utility functions satisfy $U'' < 0$, $U''' < 0$, $U'''' < 0$. These include all ruin-averse travelers who are also insatiated ($U' < 0$) and risk-averse ($U'' < 0$) no matter which utility function they adopt.

A ruin-averse traveler must be risk-averse, but not vice versa. In Example 1, if Paths 1, 2 and 3 form the choice set, risk-averse travelers may select Path 2 or 3 depending on their specific utility functions. However, as Path 3 dominates Path 2 in the third order (cf. Example 1), ruin-averse travelers will always prefer Path 3 to Path 2.

3. Admissible paths under stochastic dominance

All the three SD rules can only impose a partial order, because the dominance relationship may not exist between a pair of alternatives. Thus, the main utility of these rules is to eliminate paths that are dominated by others. The paths that are not dominated are called admissible paths in this paper. These admissible sets are useful because they provide a basis for further decision-making. Before we formally define admissible paths and discuss their properties, let us first introduce the following notation for the expository convenience. Consider a directed and connected network $G(V, A, P)$ consisting of a set of nodes

$$E[U(X)] = E[U(E[X]) + U'(E[X]) \cdot (X - E[X]) + \frac{U''(E[X])}{2!} \cdot (X - E[X])^2 + \frac{U'''(E[X])}{3!} \cdot (X - E[X])^3]$$

(1)

Recalling that the skewness is measured by $E[(X - E[X])^3]/r^3$, where $r$ is the standard deviation, $U'' < 0$ indicates that an expected-utility-maximization decision maker would always prefer negative skewness (or ruin aversion), everything else equal. According to Theorem 1, if $X$ dominates $Y$ in the third order, it implies that $X$ is preferred to $Y$ by all travelers whose utility functions satisfy $U'' < 0$, $U''' < 0$, $U'''' < 0$. These include all ruin-averse travelers who are also insatiated ($U' < 0$) and risk-averse ($U'' < 0$) no matter which utility function they adopt.

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Table 1

<table>
<thead>
<tr>
<th>Time</th>
<th>Path</th>
<th>FSD: cumulative prob.</th>
<th>SSD: area under CDF</th>
<th>TSD: integral of areas</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1 2 3 4</td>
<td>1 2 3 4</td>
<td>1 2 3 4</td>
</tr>
<tr>
<td>0</td>
<td>0.00</td>
<td>0.00 0.00 0.00 0.00</td>
<td>4.98 5.20 5.48 3.98</td>
<td>33.19 35.22 36.66 27.98</td>
</tr>
<tr>
<td>1</td>
<td>0.10</td>
<td>0.05 0.15 0.05 0.06</td>
<td>4.93 5.18 5.41 3.97</td>
<td>28.24 30.03 31.22 24.01</td>
</tr>
<tr>
<td>2</td>
<td>0.19</td>
<td>0.20 0.08 0.06 0.07</td>
<td>4.79 5.05 5.23 3.92</td>
<td>23.38 24.92 25.90 20.06</td>
</tr>
<tr>
<td>3</td>
<td>0.30</td>
<td>0.38 0.25 0.20 0.20</td>
<td>4.54 4.76 5.01 3.78</td>
<td>18.72 20.02 20.78 16.21</td>
</tr>
<tr>
<td>4</td>
<td>0.39</td>
<td>0.40 0.35 0.30 0.30</td>
<td>4.20 4.37 4.71 3.53</td>
<td>14.35 15.45 15.93 12.56</td>
</tr>
<tr>
<td>5</td>
<td>0.50</td>
<td>0.42 0.63 0.38 0.38</td>
<td>3.75 3.96 4.22 3.19</td>
<td>10.38 11.29 11.47 9.20</td>
</tr>
<tr>
<td>6</td>
<td>0.60</td>
<td>0.50 0.69 0.45 0.45</td>
<td>3.20 3.50 3.56 2.78</td>
<td>6.90 7.56 7.58 6.21</td>
</tr>
<tr>
<td>7</td>
<td>0.70</td>
<td>0.81 0.75 0.55 0.55</td>
<td>2.55 2.85 2.84 2.28</td>
<td>4.03 4.38 4.39 3.69</td>
</tr>
<tr>
<td>8</td>
<td>0.80</td>
<td>0.95 0.97 0.70 0.70</td>
<td>1.80 1.97 1.98 1.65</td>
<td>1.85 1.98 1.98 1.73</td>
</tr>
<tr>
<td>9</td>
<td>0.90</td>
<td>0.99 0.99 0.80 0.80</td>
<td>0.95 1.00 1.00 0.90</td>
<td>0.48 0.50 0.50 0.45</td>
</tr>
<tr>
<td>10</td>
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<td>1.00 1.00 1.00 1.00</td>
<td>0.00 0.00 0.00 0.00</td>
<td>0.00 0.00 0.00 0.00</td>
</tr>
</tbody>
</table>

Notes: FSD is determined according to CDF (Columns 2–5); SSD is determined according to the area under CDF between $t$ to $T$, $\forall t$ (Columns 6–9); and TSD is determined according to the integral of the areas under CDF between $t$ to $T$, $\forall t$ (Columns 10–13).
Definition 4. **FSD/SSD/TSD-admissible paths** A path \( k^* \) is an FSD/SSD/TSD-admissible path if and only if no such a path \( l^* \in K^* \) exists that \( \pi_{ij}^{l^*} > 1/\gamma > 2/\gamma > 3/\gamma \) for all \( i,j \in \mathcal{N} \).

**Theorem 1** and **Definition 4** lead to the following corollary.

**Corollary 1.** The optimal path for any insatiable/risk-averse/ruin-averse traveler must be an FSD/SSD/TSD-admissible path. However, an FSD/SSD/TSD-admissible path may not be optimal for any insatiable/risk-averse/ruin-averse traveler.

To see why the second statement in **Corollary 1** is true, note that a path may not be preferred by any traveler \( i \), because the traveler \( i \) may always find a set of paths \( A(i) \) that provides a better expected utility. However, as long as \( \cap A(i)_{vi} = \emptyset \), the path is still SSD-admissible.

Let \( I_{\text{FSD}}^r \), \( I_{\text{SSD}}^r \) and \( I_{\text{TSD}}^r \) be the sets of all FSD, SSD and TSD-admissible paths between OD pair \((r, s)\), respectively. The relationship between these sets can be shown as follows.

**Proposition 1.** \( I_{\text{FSD}}^r \subseteq I_{\text{SSD}}^r \subseteq I_{\text{TSD}}^r \).

**Proof.** To show \( I_{\text{SSD}}^r \subseteq I_{\text{FSD}}^r \), consider two paths \( k^* \) and \( l^* \). According to **Definitions 1** and 2, \( \pi_{ij}^{k^*} > 1/\gamma > 2/\gamma > 3/\gamma \). Thus, if a path is dominated in the first order, it must be dominated in the second order as well; Conversely, if a path is not dominated in the second order, it must not be dominated in the first order. That is, an SSD-admissible path must be FSD-admissible. The relationships can be proven similarly. The relationship also follows from **Theorem 1**, using the properties of the utility functions.

Consider **Example 1** again. According to the analysis, all paths except Path 4 are FSD-admissible; Paths 2 and 3 are SSD-admissible; and Path 3 is the only TSD-admissible path. Furthermore, Path 1 gives an example where admissibility does not warrant optimality. Note that although Path 1 is FSD-admissible, it does not give the least time budget for any cumulative probability (cf. the CDF data in Columns 2–5 in **Table 1**). In other words, no insatiable traveler would choose Path 1, although it is FSD-admissible by definition.

### 4. SD-admissible paths and other route choice models

We now proceed to examine how the three classes of reliability-based route choice models reviewed in Section 1.1 are related to the SD-admissible paths defined in the last section. Specifically, Section 4.1 discusses the models based on on-time arrival probability/percentile travel time; Sections 4.2 and 4.3 discuss models built on the concept of travel time budget, and the models in Sections 4.4 and 4.5 are based on the expected utility theory. We use \( E(\pi_{ij}^r) \) and \( \text{Var}(\pi_{ij}^r) \) to denote the expectation and variance of the random travel time on path \( k^* \), respectively. Also, \( f_{\pi_{ij}^r}(\cdot) \) denotes the PDF of \( \pi_{ij}^r \). \( \rho \) is used to denote the optimal reliability index.

**4.1. On-time arrival probability/percentile travel time**

A straightforward and commonly adopted reliability index is the percentile travel time (PTT) or on-time arrival probability (OTP). Mathematically,
the on-time arrival probability

4.3. Mean excess travel time

where \( E(\pi^a_k) \) is the \( a \)-percentile travel time, and the reliability index \( \rho_{\text{PTT}}(x) \) is the minimum percentile travel time. It is well-known that minimizing percentile travel time equals maximizing on-time arrival probability for a given travel time \( b \), i.e.,

\[
\rho_{\text{PTT}}(b) = \max \{ u^a_b(b), \quad \forall k^a \in K^a \} \quad (3)
\]

where \( u^a_b(b) \) is the cumulative probability at \( b \). Nie and Wu (2009a) showed that an optimal path with respect to PTT or OTP must be FSD-admissible, but the reverse statement is generally not true. For example, as mentioned in the end of Section 3, Path 1 is an FSD-admissible path but not a PTT or OTP-optimal path, because it does not give the least time budget for any cumulative probability. Thus, PTT-optimal or OTP-optimal paths can be obtained by first generating all FSD-admissible paths and then ranking the paths using (2) or (3).

4.2. Effective travel time/travel time budget (TTB)

The effective travel time, or travel time budget (TTB), is the sum of mean travel time and a safety margin, which is reserved to hedge against uncertainty. Most often, the optimal reliability index based on TTB is defined as

\[
\rho_{\text{TTB}}(\lambda) = \min \left\{ E(\pi^a_k) + \lambda \sqrt{\text{Var}(\pi^a_k)}, \quad \forall k^a \in K^a \right\} \quad (4)
\]

where \( E(\pi^a_k) + \lambda \sqrt{\text{Var}(\pi^a_k)} \) is the effective travel time for path \( k^a \), and \( \lambda \) reflects travelers’ preference for punctuality (Lo and Tung, 2003).

If each \( \pi^a_k \) follows the same type of distribution such that the same \( \lambda \) always corresponds to an identical cumulative probability \( x \) (e.g. normal distribution), the TTB index is closely related to the PTT index. Specifically, TTB can be redefined using the on-time arrival probability \( x \) as follows:

\[
\rho_{\text{TTB}}(x) = \min \left\{ E(\pi^a_k) + \lambda(x) \sqrt{\text{Var}(\pi^a_k)}, \quad \forall k^a \in K^a \right\} \quad (5)
\]

where \( E(\pi^a_k) + \lambda(x) \sqrt{\text{Var}(\pi^a_k)} = u^a_k(x) \), i.e., \( x \)-percentile travel time. Consequently, the path with the minimum TTB can also be found from all FSD-admissible paths.

With the assumption of normal distribution, \( \lambda > 0 \rightarrow x > 0.5, \quad \lambda = 0 \rightarrow x = 0.5, \) and \( \lambda < 0 \rightarrow x < 0.5 \). It is tempting to characterize the travelers with positive \( \lambda \) as “risk-averse” because they tend to reserve a positive safety margin, proportional to the standard deviation of the travel time. Similarly, negative \( \lambda \) is linked to “risk-prone”. We postulate that such an interpretation is inconsistent with the SD theory. In Example 1 described in Section 2.1, if Paths 1 and 2 form a complete choice set, Path 1 may not be preferred by any risk-averse traveler even if it provides the least TTB given 60% on-time arrival probability, because Path 1 is dominated by Path 2 in the second order (i.e., Path 1 is not SSD-admissible). Therefore, that a traveler would like to reserve a positive safety margin does not necessarily imply that he/she is risk-averse. Reserving a positive safety margin entails a strong desire for punctual arrival, which may be more properly related to the purpose of a trip instead of the risk-taking preference of a traveler.

In the general case where each \( \pi^a_k \) may follow different types of distributions, the TTB index may not be converted to a PTT index because of the lack of one-to-one correspondence between \( x \) and \( \lambda \). More importantly, minimum TTB may not be found from all FSD-admissible paths. To see this, consider the following example.

Example 2. Let \( X \) and \( Y \) be the random travel times of Paths 1 and 2, which are the only paths in consideration. Let \( F_X(t) = t, \quad t \in [0, 1] \) and \( F_Y(t) = t^2, \quad t \in [0, 1] \) be the CDFs of \( X \) and \( Y \), respectively. The mean and variance of \( X \) and \( Y \) are \( E(X) = 1/2, \quad \text{Var}(X) = 1/12 \), and \( E(Y) = 2/3, \quad \text{Var}(Y) = 1/18 \).

Note that in this example, \( X \succ Y \) by Definition 1. Hence, only Path 1 is FSD-admissible. However, Path 2 may have a smaller TTB when

\[
E(X) + \lambda \text{Var}(X) > E(Y) + \lambda \text{Var}(Y) \Rightarrow \lambda > 6
\]

That is to say, when \( \lambda > 6 \), the optimal path identified from FSD-admissible path set (i.e., Path 1) does not give the minimum TTB.

4.3. Mean excess travel time

Mean excess travel time (METT) is an extension to the percentile travel time, which according to Zhou and Chen (2008), is intended to capture the “unreliability impacts of excessively late trips”. The reliability index corresponding to METT is defined as the sum of the PTT and a tardy time measure:

\[3\] This assumption is not as restrictive as it sounds. Note that the central limit theorem may not be used to show that a route travel time follows the normal distribution approximately, because they are the sum of a number of random link travel times.
Proposition 2. There always exists a path $l^s \in \Gamma_{\text{FSD}}$ such that for a given $\alpha$, the optimal reliability index

$$
\rho_{\text{METT}}(\alpha) = v_{l^s}(\alpha) + \frac{1}{1 - \alpha} \int_{v_{l^s}(\alpha)}^{T} \left[ x - v_{l^s}(\alpha) \right] f_{l^s}(x) \, dx
$$

The second term in Eq. (6) is an expected excessive delay conditional on the choice of PTT (or the on-time arrival probability $\alpha$). The same index is often known as conditional tail expectation (CTE) in finance. Clearly, METT is meant to be more conservative than PTT, that is $\rho_{\text{METT}}(\alpha) \geq \rho_{\text{PTT}}(\alpha)$. Using integral by part and referring to Fig. 3, we have

$$
\int_{v_{l^s}(\alpha)}^{T} \left[ x - v_{l^s}(\alpha) \right] f_{l^s}(x) \, dx = xv_{l^s}(\alpha)\bigg|_{v_{l^s}(\alpha)}^{T} - \int_{v_{l^s}(\alpha)}^{T} \left( u_{l^s}(t) - v_{l^s}(\alpha) \right) (1 - u_{l^s}(t)) \, dt
$$

$$
= (\text{Area}_{\text{ODBT}} - \text{Area}_{\text{OECF}}) = \text{Area}_{\text{FTBC}} - \text{Area}_{\text{EDAC}} = \text{Area}_{\text{ABC}}
$$

where $\text{Area}_{\text{ODBT}}$ represents the rectangular area circled by O, D, B and T in Fig. 3.

We have the following result about the relationship between the minimum METT path and the FSD-admissible path set.

**Proposition 2.** There always exists a path $l^s \in \Gamma_{\text{FSD}}$ such that for a given $\alpha$, the optimal reliability index

$$
\rho_{\text{METT}}(\alpha) = v_{l^s}(\alpha) + \frac{1}{1 - \alpha} \int_{v_{l^s}(\alpha)}^{T} \left[ x - v_{l^s}(\alpha) \right] f_{l^s}(x) \, dx
$$

**Proof.** Suppose that $k^n$ is not an FSD-admissible path and let $\rho_{\text{METT}}(k^n, \alpha)$ be its METT value for a given on-time arrival probability $\alpha$. We need to show that there always exists a path $l^s \in \Gamma_{\text{FSD}}$ such that $\rho_{\text{METT}}(k^n, \alpha) \geq \rho_{\text{METT}}(l^s, \alpha)$. To prove this by contradiction, suppose $\rho_{\text{METT}}(k^n, \alpha) < \rho_{\text{METT}}(l^s, \alpha)$, $\forall l^s \in \Gamma_{\text{FSD}}$. Since path $k^n$ is not FSD-admissible, there must exist a path $l^s \in \Gamma_{\text{FSD}}$ such that $l^s > k^n$, which implies $v_{l^s}(\alpha) \leq v_{k^n}(\alpha)$, $\forall \alpha$, and $v_{l^s}(\alpha) < v_{k^n}(\alpha)$ for at least one $\alpha$. According to Eq. (6), we have

$$
\rho_{\text{METT}}(k^n, \alpha) - \rho_{\text{METT}}(l^s, \alpha) = v_{l^s}(\alpha) - v_{k^n}(\alpha) + \frac{1}{1 - \alpha} \left( \int_{v_{l^s}(\alpha)}^{T} \left[ x - v_{l^s}(\alpha) \right] f_{l^s}(x) \, dx - \int_{v_{k^n}(\alpha)}^{T} \left[ x - v_{k^n}(\alpha) \right] f_{k^n}(x) \, dx \right)
$$

**Fig. 4.** Illustration for Proof of Proposition 2.
We consider the following two cases.

Case I: \( t^R_1(x) = t^R_2(x) \). As illustrated in Fig. 4a, the arc EDC always lies below curve EC since \( F^R > 1_k^R \). According to Eq. (7)

\[
\text{Area}_{AEC} = \int_{x_1^R}^{x_2^R} [x - t^R_1(x)] f^R_1(x) dx, \quad \text{Area}_{AEDC} = \int_{x_1^R}^{x_2^R} [x - t^R_2(x)] f^R_2(x) dx
\]

Clearly, \( \text{Area}_{AEDC} > \text{Area}_{AEC} \), it follows from (8) that \( \rho_{\text{METT}}(k^R, x_0) > \rho_{\text{METT}}(l^R, x_0) \). This is a contradiction.

Case II: \( t^R_1(x) < t^R_2(x) \). As illustrated in Fig. 4b, according to Eq. (8),

\[
\rho_{\text{METT}}(k^R, x) - \rho_{\text{METT}}(l^R, x) = |DE| + \frac{1}{2} (\text{Area}_{BCD} - \text{Area}_{ACE}) = \frac{1}{2} (\text{Area}_{ADE} + \text{Area}_{BCD} - \text{Area}_{ACE}) = \frac{1}{2} (\text{Area}_{CED}) > 0
\]

Also a contradiction. This completes the proof. □

Proposition 2 implies that a minimum METT path can be identified by evaluating METT of all FSD-admissible paths. Also, the excess travel time can be computed using Eq. (7), without making any assumptions about the distributions of the path travel times. It is worth noting that more than one path could have the same and minimum METT. When this occurs, not all these paths are necessarily FSD-admissible. For example, see Fig. 3 where the CDFs of Paths 1 and 2 overlap when \( t > t^R(x) \). Hence, both paths have the same METT value for the \( x \), but Path 2 is dominated by Path 1 in the first order. Nevertheless, excluding those non-FSD-admissible paths from consideration is an acceptable choice because it does not compromise optimality.

4.4. Special utility functions

In this section, we consider three special utility functions that have been used in the transportation literature to model travelers’ route choice. Yin et al. (2004) and Mirchandani and Soroush (1987) employed a quadratic utility function (QUF) equivalent to the following:

\[
U_{\text{QUF}}(x) = \beta_0 + \beta_1 x + \beta_2 x^2, \quad \beta_1 < 0, \quad \beta_2 < 0
\]

Mirchandani and Soroush (1987) also discussed linear utility functions (LUF) and exponential utility functions (EUF), defined as:

\[
U_{\text{LUF}}(x) = \beta_0 + \beta_1 x, \quad \beta_1 < 0
\]

\[
U_{\text{EUF}}(x) = \beta_1 \exp(\beta_2 x + \beta_0), \quad \beta_1 < 0, \quad \beta_2 > 0
\]

Accordingly, the reliability indexes used in route choice are the maximum expected utilities corresponding to each utility function, namely,

\[
\rho_{\text{QUF}} = \max E[U_{\text{QUF}}(\pi^R_k)] = \max \left\{ \beta_0 + \beta_1 E(\pi^R_k) + \beta_2 \left( E(\pi^R_k)^2 + \text{Var}(\pi^R_k) \right), \quad \forall k^R \in K^R \right\}
\]

\[
\rho_{\text{LUF}} = \max E[U_{\text{LUF}}(\pi^R_k)] = \max \left\{ \beta_0 + \beta_1 E(\pi^R_k), \quad \forall k^R \in K^R \right\}
\]

\[
\rho_{\text{EUF}} = \max E[U_{\text{EUF}}(\pi^R_k)] = \max \left\{ \int [\beta_1 \exp(\beta_2 x + \beta_0)] f^R_k(x) dx, \quad \forall k^R \in K^R \right\}
\]

Then we have the following proposition.

Proposition 3. A traveler with a quadratic utility function of form (9) is risk-averse but not ruin-averse, and will always select a path that is SSD-admissible; a traveler with a linear utility function of form (10) is risk-neutral, and will always select the path with the least expected travel time; a traveler with an exponential utility function of form (11) is ruin-averse, and will always select a path that is TSD-admissible.

Proof. For any QUF as defined in (9), \( U'_{\text{QUF}}(x) = 2\beta_2 x + \beta_1 < 0, U''_{\text{QUF}}(x) = 2\beta_2 < 0, U'''_{\text{QUF}}(x) \equiv 0, \quad \forall x \geq 0 \). Thus, such a QUF is always decreasing and concave, and corresponds to risk-averse behavior (concavity implies risk aversion by Jensen’s inequality, as shown before). The optimal path for such a traveler is always SSD-admissible follows from Corollary 1. For an LUF of form (10), it directly follows from (13) that the reliability index is actually the expected travel time. Also, \( U'_{\text{LUF}} \equiv 0 \) implies that travel time variance has no impact on route choice, which is interpreted as risk-neutral. For any EUF of form (11), it is easy to verify that \( U' < 0, U'' < 0, U'' < 0, \forall x > 0 \). Recalling that \( U''' < 0 \) implies ruin aversion (cf. Eq. (1)) and Corollary 1 ensures that such travelers always choose TSD-admissible paths. □

4 Both Yin et al. (2004) and Mirchandani and Soroush (1987) considered increasing disutility functions. Thus, in their functions, \( \beta_1 > 0, \beta_2 > 0 \). Note that for disutility functions, risk aversion corresponds to convexity instead of concavity.
For LUF and EUF, finding the path with maximum expected utility can be reduced to a standard shortest path problem and solved by the dynamic programming (e.g., Loui, 1983). In fact, linear and exponential functions are probably the most important two classes among those known to possess this desirable feature. When the utility function takes the quadratic form (9), for example, standard shortest path algorithms are no longer applicable, and therefore, path enumeration seems unavoidable to find the optimal paths. According to Proposition 3, however, only the SSD-admissible paths have to be enumerated in order to find the optimal QUF paths. The set of SSD-admissible paths is potentially much smaller than the complete path set and hence is easier to enumerate.

As a side note, the optimal LUF path is clearly FSD-admissible, because the linear utility function (10) is decreasing. What is less obvious is that the maximum LUF utility (i.e., the least expected travel time) can always be achieved by an SSD-admissible path. To show this result, we need the following lemma.

**Lemma 1.** *X > Y if and only if E(X−η) ≤ E(Y−η), ∀ η < T and E(X−η) < E(Y−η), for at least one η < T, where 0 ≤ X, Y ≤ T < ∞ and X∗ = max(0, X).*

**Proof.** See Dentcheva and Ruszczyński (2003).

Given two paths C and P, it follows from the lemma that σ2(C)σ2(P) − E(σ2(C)) ≤ E(σ2(P)). Suppose path P has the least expected travel time (LET), but is not SSD-admissible. Let C be the path that dominates P in the second order, we have E(σ2(C)) ≤ E(σ2(P)). Thus, C and P must both be LET paths. Nevertheless, an LET path may not always be found from the TSD-admissible path set, as shown in Lemma 3 in Appendix B.

### 4.5. Mean–variance rule

Eq. (12) shows that, for any specific quadratic function of the form (9), the maximum expected utility can be explicitly evaluated from the mean and variance of the path travel time. Since finding such paths require path enumeration, the mean–variance rule is used to enumerate only the non-dominated paths. In this context, a path is non-dominated if no such a path exists whose mean and variance are both smaller. We denote a set of non-dominated paths based on the mean–variance trade-off as C MV.

The problem of finding C MV can be formulated as a bi-criteria shortest path problem, in which the objective is to minimize both mean and variance of the path travel time (Loui, 1983). Once C MV is determined, the optimal path can be obtained from this set based on specific utility functions, using the maximum expected utility as the reliability index. Our analysis of QUF in the previous section suggests an alternative approach. Namely, instead of using the mean–variance rule, the SSD rule can be employed to generate an SSD-admissible path set C SSD. The following result shows that using the SSD-admissible path set yields the same optimal solution as the mean–variance rule when the utility function is of the form (9).

**Proposition 4.** If a traveler’s utility function is of the form (9), the path with maximum expected utility obtained from C MV must belong to C SSD.

**Proof.** Let k* ∈ C MV and k* maximizes travelers’ expected utility for the function (9). Suppose k* ≠ C SSD. Then according to Definition 2, there must exist a path l such that the expected utility of l is larger than that of k* for all decreasing and concave utility functions. Note that function (9) is decreasing and concave, then the expected utility of l must be larger than that of k*, which contradicts with the assumption that k* gives the maximum utility.

It is worth noting that C MV and C SSD are actually two different sets, even though they share a subset that are optimal with respect to (12). Consider k* ≠ C MV. That is, there exists a path P whose mean and variance are both smaller. However, it is possible that k* ∈ C SSD, unless one can show that the same P would produce a larger expected utility for any decreasing and concave function, which clearly need not to be the case. Conversely, if k* ≠ C SSD, k* could still belong to C MV. To see this, consider Example 2 discussed before, in which X > Y and X > Y by Definitions 1 and 2. Thus, Path 2 is not SSD-admissible. However, because E(X) > E(Y) and Var(X) < Var(Y), both paths are non-dominated and should belong to C MV. In this special case, C SSD ⊂ C MV. However, the numerical experiment presented in Section 6.1 gives an example where C MV = C SSD.

### 4.6. Summary

We close the section by providing a complete picture of how the optimal path sets generated from different definitions of reliability indexes are related to SD-admissible path sets and to each other. We first define these sets for the ease of reference. Let

(i) I FSD, I SSD and I TSD be the sets of FSD- , SSD- and TSD-admissible paths, respectively;
(ii) I TTB be the set of paths that have the minimum travel time budget (TTB) for at least one punctuality parameter λ (cf. (4)), and I PTT be the set of paths that have the minimum percentile travel time (PTT) for at least one on-time arrival probability α;
(iii) $\Gamma_{\text{METT}}$ be the set of paths that are optimal with respect to METT for at least one $a$;
(iv) $\Gamma_{\text{MV}}$ be the non-dominated path set under the mean–variance rule; and
(v) $\Gamma_{\text{EUF}}, \Gamma_{\text{LUF}}$ and $\Gamma_{\text{QUF}}$ be the sets of paths that are optimal in terms of expected utility for at least one exponential, linear and quadratic utility function of the forms (11), (10) and (9) respectively;

The relationships between these route sets are depicted by Fig. 5. First of all, note that all plots show $\Gamma_{\text{rs}}^{\text{TSD}} \subseteq \Gamma_{\text{rs}}^{\text{SSD}} \subseteq \Gamma_{\text{rs}}^{\text{FSD}}$, as asserted in Proposition 1. Plot (a) shows that any optimal path under PTT must be always FSD-admissible, while the optimal path under TTB may be not. Plot (b) highlights the fact that the non-dominated paths under the mean–variance rule may be neither SSD-admissible nor FSD-admissible; it also indicates $\Gamma_{\text{MV}}$ and $\Gamma_{\text{FSD}}$ share a common subset $\Gamma_{\text{QUF}}$ (Proposition 4). Plot (c) shows that both LUF- and QUF-optimal paths must be SSD-admissible; while EUF-optimal paths must be TSD-admissible. Plot (d) indicates the $\Gamma_{\text{METT}}$ is “almost” contained in $\Gamma_{\text{FSD}}$, in the sense that at least one METT-optimal path can be found from FSD-admissible path set for any given $a$ (Proposition 2).

To summarize, the paths that optimize a wide variety of reliability indexes can be found from an SD-admissible path set. Therefore, the problem of finding these optimal paths can be decomposed into two steps: generating admissible paths under an appropriate SD rule; and evaluating the desired reliability index over all admissible paths to find the optimal solution. By applying the SD rule in a general dynamic programming (GDP) framework, the first step in this proposed solution framework can be performed much more efficiently than enumerating all paths, which is not even a viable option in most cases. To the details of that critical step we now turn.

5. Finding SD-admissible paths

The problem of finding FSD-admissible paths is solved in Nie and Wu (2009a), with a label-correcting algorithm that operates on the principle of general dynamic programming (GDP). This section extends their algorithm to the higher order SD, by proving that GDP still applies in these cases. Before we present the algorithm, we first discuss the interesting properties of the so-called Pareto frontier, which is the upper envelop of the CDFs for all admissible paths.

5.1. Pareto frontier

In this section we assume all travelers adopt PTT as the reliability index for route choice. For any OD pair $(r, s)$, define the Pareto frontier as

![Fig. 5. Relationship between SD-admissible path sets and the solution path sets of other route choice models.](image)
\[
u_D^\alpha(b) = \max \{ u_k^\alpha(b), \quad \forall k \in \Gamma_D^{\alpha} \} \tag{15}\]

where \( u_k^\alpha(\cdot) \) is the CDF of the travel time on path \( k^{\alpha} \), \( D = \text{FSD, SSD, TSD} \). Specifically, \( u_{\text{FSD}}^\alpha \), \( u_{\text{SSD}}^\alpha \) and \( u_{\text{TSD}}^\alpha \) are called the FSD, SSD and TSD Pareto frontier. \( u_D^\alpha(b) \) provides the maximum on-time arrival probability for time budget \( b \) under an SD rule \( D \). Pareto frontiers can also be represented using the inverse CDF, \( v_D^\alpha(\cdot) \):

\[
v_D^\alpha(\alpha) = \min \{ v_k^\alpha(\alpha), \quad \forall k \in \Gamma_D^{\alpha} \}, \quad D = \text{FSD, SSD, TSD} \tag{16}\]

\( v_D^\alpha(\cdot) \) provides the minimum percentile travel time for a given on-time arrival probability \( \alpha \) under an SD rule \( D \).

The second inequality indicates that for a given on-time arrival probability \( \alpha \), a risk-averse traveler may reserve more travel time than an insatiable traveler without special risk-preference, and a ruin-averse traveler may budget more travel time than a risk-averse traveler. Such a discrepancy may be interpreted as a risk (ruin) premium or willingness to pay to avoid risk (ruin).

For an illustrative example, see Fig. 6, where all three paths are FSD-admissible, and only Paths 2 and 3 are SSD-admissible. Suppose a specific risk-averse traveler prefers Path 3 to Path 2 for the utility function he/she adopts. For an on-time arrival probability \( \alpha \), \( t_1 \), \( t_2 \) and \( t_3 \) are the percentile travel time given by Paths 1, 2 and 3, respectively, and \( t_1 < t_2 < t_3 \). A non-risk-averse traveler only need to budget \( t_1 \) for travel in order to ensure the probability \( \alpha \). However, the aforementioned risk-averse traveler is willing to pay extra travel time \( t_3 - t_1 \) to avoid risks. That is, \( t_3 - t_1 \) is this traveler’s willingness-to-pay for risk aversion.

The discrepancy between the FSD and SSD Pareto frontiers provides a lower bound for the risk premium of all risk-averse travelers since for any path \( k^{\alpha} \in \Gamma_{\text{SSD}}^{\alpha} \)

\[
0 \leq v_{\text{SSD}}^\alpha(\alpha) - v_{\text{FSD}}^\alpha(\alpha) \leq u_{\text{FSD}}^\alpha(\alpha) - u_{\text{SSD}}^\alpha(\alpha), \quad \forall \alpha \in [0, 1) \tag{17}\]

The gap between the FSD and SSD frontiers at \( \alpha \) in Fig. 6 is \( t_2 - t_1 \), which is smaller than \( t_3 - t_1 \). Similarly, the gap between the SSD and TSD frontiers reflects the lower-bound for a ruin-averse traveler’s willingness-to-pay for ruin aversion.

5.2. Solution algorithm

SD-admissible paths can be found by checking the stochastic dominance relationship between any pair of paths and eliminating those that are dominated. However, since the problem is NP-hard, such a brute-force method may not be computationally feasible. SD-admissible paths have two important properties that make it possible to greatly improve the efficiency of the search process. The following result can be viewed as an extension to those given in Nie and Wu (2009a).

**Proposition 5.** FSD/SSD/TSD-admissible paths have the following properties: (1) They must be acyclic; (2) Subpaths of any FSD/SSD/TSD-admissible paths must also be FSD/SSD/TSD-admissible.

**Proof.** See Appendix B. \( \square \)

---

**Fig. 6.** Illustration of risk premium.
Acyclicity ensures that the number of admissible paths in a general network is finite; and the second property ensures the applicability of the Bellman’s principle of optimality. Consequently, any of the three SD admissible path sets can be constructed recursively using label-correcting (LC) algorithms. A brief description of a generic form of the algorithm is given below. Note that the algorithm always find all-to-one admissible paths.

Algorithm SD-LC

**Step 0** Initialization. Let $$0^a$$ be a dummy path from the destination to itself. Initialize the scan list $$Q = \{0^a\}$$. set $$\pi^{0}_0 = 1$$ with probability 1.

**Step 1** Select the first path from $$Q$$, denoted as $$l^a$$, and delete it from $$Q$$.

**Step 2** For any predecessor node $$i$$ of $$j$$, create a new path $$k^a$$ by extending $$l^a$$ along link $$ij$$.

**Step 2.1** Calculate the distribution of $$\pi^{k}_i$$ from the distribution of $$\pi^{l}_i$$ by convolution.

**Step 2.2** Compare the distribution of the new path to those of all existing admissible paths by the appropriate SD rule: if any of the existing path dominates $$k^a$$, drop $$k^a$$ and go back to Step 2; otherwise, delete all paths that are dominated by $$k^a$$ from $$I^a_D$$ (where $$D = \text{FSD}, \text{SSD},\text{and TSD}$$), set $$I^a_D = \{k^a\}$$, and update $$Q = Q \cup \{k^a\}$$.

**Step 3** If $$Q$$ is empty, stop; otherwise go to Step 1.

The above algorithm does not have a polynomial complexity, as shown in Miller-Hooks (1997) and Nie and Wu (2009a). However, existing numerical evidence suggests that FSD-admissible path sets are rather small on typical transportation networks, and that the performance of the algorithm is generally satisfactory in practice.

Two operations in the above algorithm are worth a bit more explanation. First, in step 2.1, the distribution of the new path is computed by convolving the distributions of the subpath travel time and the link travel time. That is, we do not rely on the central limit theorem to approximate the summation of the distributions. As convolution is a computationally intensive procedure, an efficient implementation is critical to the performance of the algorithm. The reader is referred to Nie et al. (submitted for publication) and Wu and Nie (submitted for publication) for recent developments on computing convolutions.

Another time-consuming procedure has to do with determining stochastic dominance among paired paths. For FSD, Definition 1 can be directly used to check dominance since the CDFs of the path travel time distributions have to be computed and stored anyway. For SSD and TSD, however, using Definitions 2 and 3 are neither convenient nor efficient, as it involves numerical integrations. As an alternative, Lemma 1 can be employed to check dominance in the case of SSD, which only requires evaluating $$E(\pi^{k}_i - \eta)_+$$ for a discrete set of $$\eta$$. Similarly, for the case of TSD, we have the following result.

**Lemma 2.** If and only if $$E[E(X - \eta)_+ - \gamma]_+ \leq E[E(Y - \eta)_+ - \gamma]_+$$ for at least one $$0 \leq \gamma \leq T$$, where $$0 = X, Y \leq T < \infty$$.

**Proof.** See Appendix B. □

With Lemma 2, the dominance in TSD can be determined by evaluating $$E[E(\pi^{k}_i - \eta)_+ - \gamma]_+$$ for sets of discrete values for $$\eta$$ and $$\gamma$$.

6. Numerical experiments

A small example is first tested to verify the relationship between the SD-admissible paths and the optimal path sets based on other reliability indexes. Then, the label-correcting algorithm presented in Section 5 is tested on two real road networks from the Chicago area to demonstrate that finding SD-admissible paths is computationally feasible even on fairly large networks. Finally, we gave several examples where SD-admissible paths are used to find optimal paths defined by different reliability indexes. The algorithm was coded using C++ and tested on a Windows-XP(64) workstation with two 3.00 GHz Xeron CPUs and 8 G RAM.

6.1. Relationship between different route choice models

To verify the analysis presented in Section 4, a small network with 24 nodes and 37 links (see Fig. 7) is employed. Links in the network are categorized into three classes: expressways, arterial roads and local streets, depending on the means and variances of the travel time on them, as shown in Table 2. The mean and variance of the travel time on a link are determined from Table 2, except for Links 1–2 and 20–18. The former’s mean and variance are four times of the standard values for arterial streets and the latter’s mean and variance are 1.8 times of those for expressways. Link travel times are assumed to follow gamma distributions. Given mean $$\mu$$ and variance $$\sigma^2$$, the gamma distribution is given by:

$$f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

with $$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$.

Travel time on freeways and arterial streets is known to closely follow a Gamma distribution (e.g., Polus, 1979), and the travel time data from loop detectors and toll transponders also confirms this assumption. Readers are referred to Nie et al. (submitted for publication) for more details.
\[ p_j(\kappa) = \frac{1}{\theta^\kappa \Gamma(\kappa)} (C_j)^{\kappa-1} e^{-(C_j)/\theta} \]  

(18)

where \( \kappa = \mu^2/\sigma^2 \), \( \theta = \sigma^2/\mu \), and \( \Gamma(\kappa) \) is gamma function about \( \kappa \).

We consider the route choice between O–D pair (1,18). As shown in Fig. 7, there are in total 26 paths between the O–D pair. The travel time distributions of these paths can be obtained by convolving distributions of link travel times. This is relatively easy to do here because the sum of two Gamma distributions still follows a Gamma distribution, with the mean and variance equal to the sum of means and variances of the two distributions respectively. Therefore, the travel time distribution of a path can also be determined by Eq. (18), as long as the means and variances are added along its member links.

Optimal paths based on all route choice models discussed in Section 4 are generated on this network, as well as the three SD-admissible path sets. The solution sets are listed in Table 3. For QUF, EUF and TTB models, we selected five different sets

---

**Table 2**

Mean and variance of travel time on three types of links.

<table>
<thead>
<tr>
<th></th>
<th>Expressway</th>
<th>Arterial road</th>
<th>Local street</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Variance</td>
<td>4</td>
<td>2.5</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 3**

Solution sets solved under various route choice model.

<table>
<thead>
<tr>
<th>Route choice model</th>
<th>Path ID</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>FSD-admissible</td>
<td>3, 8, 9, 11, 21</td>
<td></td>
</tr>
<tr>
<td>SSD-admissible</td>
<td>3, 8, 9, 21</td>
<td></td>
</tr>
<tr>
<td>TSD-admissible</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>Mean–variance rule</td>
<td>3, 8, 9, 21</td>
<td>with ( \alpha = 0.01, 0.02, \ldots, 1 )</td>
</tr>
<tr>
<td>Least expected time</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>Least variance</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>TTB-optimal</td>
<td>9, 21</td>
<td>with five different ( k ) in Eq. (4)</td>
</tr>
<tr>
<td>QUF-optimal</td>
<td>21</td>
<td>With five different sets of ( b ) in Eq. (12)</td>
</tr>
<tr>
<td>EUF-optimal</td>
<td>21</td>
<td>With five different sets of ( b ) in Eq. (14)</td>
</tr>
</tbody>
</table>

---
of function parameters and enumerate all identified optimal paths. The results indicate that the TSD-admissible path set is a subset of the SSD-admissible set, which is a subset of the FSD-admissible path set, as asserted in Proposition 1. Interestingly, the FSD and SSD Pareto frontiers are almost identical, as shown in Fig. 8.

Path 21 is an exclusive expressway path, and by the design of the experiment, it is the path that has the least expected travel time (LET). When the desired on-time arrival probability \( a \) is less than 93%, Path 21 is also the optimal path whether the traveler is insatiable (FSD), risk-averse (SSD) or ruin-averse (TSD). However, when 93% \( \leq a \leq 99% \), Path 9 becomes the optimal path for insatiable and risk-averse travelers. Note that on Path 3, a local street takes the place of an arterial link on Path 9, which makes it a more attractive option as the required reliability becomes still higher. On the other hand, any ruin-averse traveler would only choose Path 21, the LET path. It might seem counter-intuitive that the most conservative group would always choose the LET path. However, a close look reveals that when \( a > 93% \), travelers who choose Path 21 actually have to budget more time for travel, compared with Path 9. Fig. 8b highlights the gap between FSD and TSD frontiers in the range of \( a \in [0.9, 1) \). For example, if the desired on-time arrival probability is 95%, the travel time required by Path 21 is 4.4% more than that of Path 9.

For all tested QUF and EUF functions, Path 21 is always the optimal path. Also, the non-dominated path set under the mean–variance rule in this experiment happens to be identical to the set of SSD-admissible paths. We note that a non-dominated path may not even be FSD-admissible, however, as shown by the counter-example given in Section 4.2. Finally, the METT-optimal paths for all \( a \) and the TTB-optimal paths solved from five different \( k \) are all FSD-admissible paths, which are consistent with our analysis in Sections 4.2 and 4.3.

6.2. SD-admissible paths on large networks

We now apply the label-correcting algorithm on two real road networks to generate three SD-admissible path sets. The first one is a sketch of the Chicago metropolitan area (CK), which has 933 nodes and 2950 links. The link travel times in the CK network are assumed to follow Gamma distributions, whose parameters are randomly selected based on the same setting as in Wu and Nie (2009). The second network is a more detailed representation of the same area, from the latest travel planning model prepared by the Chicago Metropolitan Agency for Planning (CMAP). The CMAP network has 15,037 nodes and 44,331 links. Travel time distributions on freeways and toll roads in this network are constructed from traffic data collected from loop detectors and toll transponders; the distributions on other roads are assumed to follow the Gamma distributions and the parameters are estimated from the free flow travel time and congestion level. The reader is referred to Nie et al. (submitted for publication) for details about the estimation of these parameters and other details about the CMAP network. CMAP data sets include four sets of link distributions, each corresponding to a different time period of day. In this experiment, the distributions for the morning peak period is used. In all tests presented in this section, a Pareto frontier is represented by 100 discrete points.

**Table 4**

Average computational performance of Algorithms SD-LC over ten runs in two networks.

| Network | FSD CPU time (s) | Ave. \( |I_{SD}^P| \) | Max. \( |I_{SD}^P| \) | SSD CPU time (s) | Ave. \( |I_{SD}^P| \) | Max. \( |I_{SD}^P| \) | TSD CPU time (s) | Ave. \( |I_{SD}^P| \) | Max. \( |I_{SD}^P| \) |
|---------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| CK      | 0.819           | 2.341           | 10.200          | 0.711           | 1.719           | 5.200           | 0.475           | 1.559           | 2               |
| CMAP    | 162.57          | 2.27            | 42.23           | 38.23           | 1.45            | 7.75            | 23.79           | 1.27            | 2               |

Fig. 8. FSD, SSD and TSD Pareto frontiers between the O–D pair (1, 18).
Ten destinations are randomly selected for each network, then the SD-LC algorithm is executed for each destination, finding all-to-one FSD-, SSD- and TSD-admissible paths. Three indicators are generated to gauge the computational performance: the CPU time, the average and maximum number of FSD-, SSD- and TSD-admissible paths. The average values of these three indicators over the ten runs are reported in Table 4. As shown in the table, it took the algorithm on average less than one second to identify all admissible paths (FSD/SSD/TSD) for the CK network. The CPU times required to find all FSD-, SSD- and TSD-admissible paths on the CMAP network is 163, 38 and 24 s on average. While these CPU times are significantly higher than those for the CK network, they are deemed acceptable especially considering the sheer size of the problem. Interestingly, it is faster to solve SD-admissible path problems in the higher order case, apparently because higher-order SD admissible path sets are significantly smaller. For the CMAP networks, solving the SSD-admissible paths consumed only one quarter of the CPU time required by the FSD counterpart.

6.3. Application of SD-admissible paths

Once generated, SD-admissible paths can be used to find optimal paths defined using various reliability indexes. In the following, three experiments are conducted using the CMAP network and data to demonstrate this idea. Due to the size of the network, these optimal solutions cannot be directly verified by path enumeration.

- **Find the METT-optimal path between O–D pair (1, 5678):** We first solve the FSD-admissible path problem: for the destination 5678, it took about 54 s to finish. There are only two FSD-admissible paths between this particular O–D pair, and we can easily identify the minimum METT as 1647.73 from these two paths.
- **Find the QUF-optimal path between O–D pair (1, 5890):** The adopted quadratic utility function is \( U(x) = 15,000 - x^2 \). In this case, the SSD-admissible path problem is first solved for the destination 5890. The computation time was about 64 s. There are two SSD-admissible paths between the O-D pair and the maximum expected utility is computed as 9448.14 from the two paths.
- **Find the EUF-optimal path between O–D pair (1, 5890):** The adopted exponential utility function is \( U(x) = -\exp(0.005x + 0.1) \). We first solve all TSD-admissible paths for the destination. The required CPU time is 40 s. Interestingly, the TSD-admissible path set for this O–D pair is exactly same as the SSD-admissible path set generated before. The maximum expected utility for this function is \(-1809.0\), but the EUF-optimal path is different from the QUF-optimal path found in the last experiment.

7. Conclusions

Route choice is affected by the concerns for travel reliability, depending on how the importance of a trip is perceived and how risk-taking references are incorporated. In this paper, travelers are assumed to set a desired on-time arrival probability (i.e., the probability of completing the trip on-time or earlier) for each trip according to its perceived importance. However, due to their different risk-taking preferences, the travelers may choose different routes and reserve different amounts of travel time for the same trip. Using the theory of stochastic dominance, this paper proposes a unified approach to model such heterogeneous risk-taking behavior in route choice. Specifically, the first-, second- and third-order stochastic dominance (FSD, SSD, TSD) are respectively linked to insatiability, risk aversion and ruin aversion within the framework of expected utility maximization. Accordingly, the paths that may be selected by travelers of different risk-taking preferences can be obtained by enumerating the corresponding SD-admissible paths. General dynamic programming can be employed to generate such SD-admissible paths, because subpaths of an SD-admissible path must be still SD-admissible, as proven in this paper.

This paper reveals interesting connections between the SD theory and several other reliability-based route choice models. Some of these connections is already known in the literature, such as the equivalence between FSD and the maximization of on-time arrival probability (or minimization of percentile travel time). New results from this paper include: (1) FSD-, SSD- and TSD-admissible paths are inclusive of any maximum-expected-utility paths based on a class of linear, quadratic and exponential utility functions, respectively; (2) any path that minimizes mean excess travel time can be found from FSD-admissible path set; (3) the paths that minimize the travel time budget or effective travel time may not be FSD-admissible unless for special distributions; and (4) the mean–variance rule may yield a non-dominated path set inconsistent with the FSD-admissible path set. These findings provide the interpretation of risk-taking behavior for these route choice models, consistent with the SD theory. More importantly, the SD-based approach provides a unified and computationally viable solution framework to reliability-based optimal path problems, i.e., identifying optimal paths from a corresponding SD-admissible path set, which is usually small and can be generated without having to enumerate all paths.

We proposed a generic label-correcting algorithm to find the FSD-, SSD-, and TSD-admissible paths. In particular, analytical results are given to simplify the determination of higher-order stochastic dominance. Numerical experiments are conducted to test the algorithm. The results verify our analysis of the relationship between SD-admissible path sets and other route choice models. Other findings from the experiments are: (1) the efficiency of the label-correcting algorithm is satisfactory, even on a large-scale regional transportation network; (2) the higher-order stochastic dominance yields less admissible paths and seems to substantially reduce the computation time; and (3) the average size of SD-admissible path sets is relatively small, which suggests that finding an optimal path from these sets based on additional conditions (such as a specific utility function) is easy.
Our analysis indicates that the route choice models based on TTB and mean–variance rule are not always compatible with the SD theory. We postulate that the SD-admissible path sets may still be used to the generate approximations to the solutions of these problems. However, further research is needed to understand which SD rule provides a better approximation and to find the bounds of the approximation errors. Another direction for future research is to apply the SD-based route choice model in traffic assignment and network design problems.

**Acknowledgements**

The authors would like to thank three anonymous reviewers for their constructive comments. This research was supported by National Science Foundation under the Award Number CMMI-0928577.

**Appendix A. Notation**

<table>
<thead>
<tr>
<th>Network</th>
</tr>
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<tbody>
<tr>
<td>$\mathcal{A}$</td>
</tr>
<tr>
<td>$\mathcal{N}$</td>
</tr>
<tr>
<td>$n$</td>
</tr>
<tr>
<td>$m$</td>
</tr>
<tr>
<td>$r, s$</td>
</tr>
<tr>
<td>$k^r$</td>
</tr>
<tr>
<td>$c_{ij}$</td>
</tr>
<tr>
<td>$T_{FSD}$, $T_{SSD}$, $T_{TSD}$</td>
</tr>
<tr>
<td>$\pi^r_k$</td>
</tr>
<tr>
<td>$\pi^r_k \triangleright ij$</td>
</tr>
</tbody>
</table>

**Probability**

- $F_X(\cdot)$: cumulative density function of random variable $X$
- $F'_X(\cdot)$: cumulative density function of travel time on link $ij$
- $p_{ij}(\cdot)$: probability density function of travel time on link $ij$
- $p_{k}^{rs}(\cdot)$: probability density function of path travel time $\pi^r_k$
- $u_{k}^{rs}(\cdot)$: cumulative density function of path travel time $\pi^r_k$
- $t_{k}^{rs}(\cdot)$: inverse function of $u_{k}^{rs}(\cdot)$
- $u_{FSD}^{rs}(b)$: maximum probability of arriving on time following an optimal routing policy for OD pair $(r, s)$, given time budget $b$ if FSD is applied
- $v_{FSD}^{rs}(x)$: minimum travel time following an optimal routing policy for OD pair $(r, s)$, given desired on-time arrival probability $x$ if FSD is applied
- $E(\pi^r_k)$, $\text{Var}(\pi^r_k)$: mean and variance of path travel time $\pi^r_k$

**Other**

- $T$: finite upper bound of the travel time support
- $U(\cdot)$: utility functions
- $\rho_A(\cdot)$: reliability index for route choice model $A$, e.g., $A$ refers to TTB, METT, EUF, QUF, LUF, and mean–variance (MV) tradeoff, etc.

**Appendix B. Proofs**

**Lemma 2.** $X \sim Z$ if and only if $E[E(X - \eta) + \gamma] \leq E[E(Y - \eta) + \gamma]$, $\forall \eta, \gamma \leq T$ and $E[E(X - \eta) + \gamma] < E[E(Y - \eta) + \gamma]$ for at least one $0 \leq \eta, \gamma < T$, where $0 < X, Y < T < \infty$.

**Proof.** Note that

$$E[X - \eta] = \int_{-\eta}^{T} (t + \eta) dF_X(t) = \int_{-\eta}^{T} t dF_X(t) - \eta \int_{-\eta}^{T} dF_X(t) = T - \eta - \int_{-\eta}^{T} F_X(t) dt$$

Then we have

$$E[E(X - \eta) + \gamma] = \int_{-\eta}^{T} \left( T - \int_{-\eta}^{T} F_X(t) dt - \gamma \right) p(\eta) d\eta = T - \int_{-\eta}^{T} \int_{-\eta}^{T} F_X(t) p(\eta) dt d\eta - \int_{-\eta}^{T} \gamma p(\eta) d\eta$$
where \( p(\eta) \) is the probability density function (PDF) of \( \eta \). Given \( \forall \eta \in \mathcal{R}, p(\eta) \) follows a uniform distribution. Therefore, \( p(\eta) \) is a constant. Let \( p(\eta) = C > 0 \), then

\[
E\left[ E(X - \eta)_+ - \gamma \right] = T - C \int \int F_X(t) dt d\eta
\]

(19)

Therefore, \( E(E(X - \eta)_+ - \gamma) \leq 0 \), \( \forall \eta, \gamma \leq T \) and \( \eta, \gamma < T \).

\[ \square \]

Lemma 3. It is not always possible to find an LET path from an TSD-admissible path set.

\[ \textbf{Proof.} \] Suppose there are two random variables \( X \) and \( Y \) such that \( Y \succ X \). Given that

\[
E(X) = \int \int t f_X(t) dt = \int t F_X(t) dt = T - \int F_X(t) dt = T + \int \int F_X(t) dt dF_X(w)\]

then

\[
E(X) - E(Y) = \int \int F_X(w) dw - \int \int F_Y(w) dw + \int \int F_Y(w) dw dt - \int \int F_X(w) dw dt
\]

\[ Y \succ X \text{ implies } \int \int F_X(w) dw - \int \int F_Y(w) dw dt \geq 0 \text{ according to Definition 3.} \]

However, the sign of \( \int \int F_X(w) dw - \int \int F_Y(w) dw dt \) cannot be determined just based on \( Y \succ X \). Therefore, it is possible that \( E(X) \leq E(Y) \). \[ \square \]

Proposition 5. (1) Any FSD/SSD/TSD-admissible paths must be acyclic. (2) Subpaths of FSD/SSD/TSD-admissible paths must be also FSD/SSD/TSD-admissible.

\[ \textbf{Proof} \]

1. \[ \text{Nie and Wu (2009a)} \] showed that a path with cycles cannot be FSD-admissible. Thus, it cannot be SSD-admissible or TSD-admissible because of Proposition 1.

2. The case for FSD-admissible path is proven in Nie and Wu (2009a). We only need to show that SSD and TSD-admissible paths have the same property here.

Consider path \( k^i \) at node \( i \) and its subpath \( k^{ij} \) at node \( j \), where \( k^{ij} = k^i \odot ij \). Suppose path \( k^i \) is SSD-admissible path at node \( i \), but \( k^{ij} \) is not SSD-admissible at node \( j \). Thus, there must exist a path \( l^j \) such that \( l^j \succ k^{ij} \).

Recalling

\[
u_i^j(b) = \int_0^b u_i^j(b - w)p_j(w) dw, \quad u_i^j(b) = \int_0^b u_i^j(b - w)p_j(w) dw.
\]

then we have

\[
\int_t^T u_i^j(w) dw = \int_t^T \int_0^w u_i^j(w - z)p_j(z) dz dw,
\]

\[
\int_t^T u_i^j(w) dw = \int_t^T \int_0^w u_i^j(w - z)p_j(z) dz dw
\]

Therefore,

\[
\int_t^T u_i^j(w) dw - \int_t^T u_i^j(w) dw
\]

\[
= \int_t^T \int_0^w u_i^j(w - z)p_j(z) dz dw - \int_t^T \int_0^w u_i^j(w - z)p_j(z) dz dw
\]

\[
= \int_0^w \left( \int_t^T u_i^j(w - z) d(w - z) \right) df_i(z)
\]

\[
- \int_0^w \left( \int_t^T u_i^j(w - z) d(w - z) \right) df_j(z)
\]
because path $P^L = 2k^L$ (i.e., $\int_0^T u^L_k(w) \, dw = \int_0^T u^L_k(w) \, dw \cdot \frac{w}{C_0}$). It implies that path $k^L$ must not be SSD-admissible. A contradiction. The case for TSD-admissible can be proven similarly and is omitted for brevity. □

References


